SOME RINGS ARE HEREDITARY RINGS

BY

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ABSTRACT

Let R be a bounded Noetherian Prime ring. The Asano-Michler theorem shows that R is a bounded Dedekind ring if every prime ideal of R is invertible. We provide a simple proof of the Asano-Michler theorem, and we suggest some possible generalizations. We also prove that if the proper residue rings of R are QF-rings then R is a bounded Dedekind ring, and generalize this result to LD-rings.

The purpose of this paper is to obtain several types of sufficient conditions for a ring to be an hereditary ring, as well as to develop some techniques of getting new projective ideals from a given set of projective ideals. Among others there results a proof for a theorem of Asano-Michler (Theorem 14) and generalizations of it (Theorems 8–9) which does not use localization [8]. The technique being one of the purposes in this paper, the reader will notice that sometimes we provide different proofs for a given result.

Specializing to the commutative case one recovers many known results (e.g [3], [6] and [11]) however the methods used here suggests some non standard proofs, the main difference seems to lie in that we do not use localizations.

A ring R is presumed to have an identity. All modules are unitary left modules and all ideals are left ideals unless otherwise specified.

An LD-ring is a left bounded ring of finite left Goldie dimension, all of whose proper residue rings are left Artinian principal ideal rings.

For the definition and properties of LD-rings we refer to [12], and for those of Goldie rings to [5] and [9].

Our first step is to obtain sufficient conditions for an ideal to be a projective left module.

Received March 18, 1971 and in revised form August 12, 1971

LEMMA 1. *Let $M_1, \cdots, M_t$ be ideals in the ring R such that:*

(1) *R/M_i is a simple artinian ring for $i = 1, \cdots, t$*

(2) *M_i is a finitely generated projective left module for $i = 1, \cdots, t$.*

Let I be a left ideal that contains a product of ideals, $A_1 \cdots A_n$, such that for every integer i, $1 \leqq i \leqq n$, there exists an integer j, $1 \leqq j \leqq t$ so that $A_i = M_j$. Then I is a left projective module.

PROOF. Obviously $R/A_1 \cdots A_n$ is a left artinian ring. Since $I \supset A_1 \cdots A_n$ there exists an ideal J in R such that $J \supset I$, and J/I is a simple module. But $(A_1 \cdots A_n)(J/I) = 0$, therefore J/I is a simple $R/A_1 \cdots A_n$ module. Consequently there exists an ideal M_k, $1 \leqq k \leqq t$, such that $M_k (J/I) = 0$.

In particular, J/I is isomorphic to a left direct summand of R/M_k. As M_k is assumed to be a left projective module, then l.p.dim $R/M_k \leqq 1$, and consequently l.p.dim $J/I \leqq 1$.

Fixing $B = A_1 \cdots A_n$, let I be a maximal ideal containing B, such that I is not a left projective module. If no such I exists we are done. Let J be a left ideal that contains I, and such that J/I is a simple module. From the maximality of I it follows that J is a left projective module, and from l.p.dim $J/I \leqq 1$ it now follows that I is a left projective module which is a contradiction. This completes the proof.

REMARK. The assumption that M_i are finitely generated left modules can be replaced by $R/A_1 \cdots A_n$ being a left artinian ring. However under the assumption that M_t are finitely generated as left modules it follows that I is a finitely generated left module.

An immediate consequence is:

COROLLARY 2. *A commutative ring, with Artinian (proper) residue rings, and maximal projective ideals is a Dedekind domain.*

A different type of condition that assures us of the projectivity of an ideal is:

LEMMA 3*. *Let M be a maximal ideal in a ring R so that $M = Rx + M^2$, x being an element of R so that:*

(i) *Rx is a projective left module.*

(ii) *$Rx \supset AM$ for some ideal A in R that is not contained in M, and for which $AM = MA$.*

If R/M is an Artinian ring, then M is a left projective module.

* Professor D. Zelinsky pointed out that the same result follows, using a similar proof, in case the existence of x is replaced by the existence of ideals A and I so that $AnM \subset I \subset M$, where I is a left projective module.

PROOF. Immediate consequences from the assumptions are: $Rx \not\supseteq A$, $M+A=R$, $MA = AM = M \cap A$, and $R/MA \simeq R/M \oplus R/A$. Therefore $R/Rx = (A + Rx)/Rx \oplus (M+Rx)/Rx$ and $(A + Rx)/Rx \neq 0$. Since $M[(A+Rx)/Rx]=0$ and R/M is an Artinian simple ring, there exists a simple left R-module S so that $(A+ Rx)/Rx$ (and R/M) is isomorphic to a direct sum of finitely many copies of S. As Rx is a projective left module we have l.p.dim $R/Rx \leqq 1$, consequently l.p.dim $(A + Rx)/Rx \leqq 1$. This implies l.p.dim $S \leqq 1$, whence l.p.dim $R/M \leqq 1$ and thus M is a left projective module.

Remark that condition (i) holds if x is a regular element in R.

We next aim at some sufficient conditions on a ring R to be an hereditary ring. Most of the results apply to bounded orders in a simple Artinian ring. We start with a ring that is not necessarily an order in a simple Artinian ring.

PROPOSITION 4. *A left bounded ring, all of whose residue rings are left Artinian rings, and all of whose maximal two sided ideals are projective left modules, is a left hereditary ring.*

PROOF. Let $I \neq 0$ be any left ideal. If I is not an essential left ideal there exists a left ideal J so that $I \oplus J$ is an essential left ideal in R. Let $K \neq 0$ be a two sided ideal that is contained in $I \oplus J$. Since R/K is a left Artinian ring, we may assume that $I \oplus J$ contains a finite product of maximal ideals of R. The conclusion is a consequence of Lemma 1.

Remark that such a ring need not be Noetherian, e.g. there exist semiprimary non Noetherian rings satisfying the assumptions of Proposition 4.

A similar result concerning orders in simple Artinian rings is:

PROPOSITION 5. *Let R be a left bounded prime ring such that R/J is a QF-ring whenever J is a non-zero two-sided ideal. Let $\bigcap_{n=1}^{\infty} M^n = 0$ for every maximal two-sided ideal M. If for every regular element x in R there exists a non-zero two sided ideal K so that $K \subset Rx$ then R is a left Noetherian, left hereditary ring.*

PROOF. If R is an Artinian ring, it is necessarily a simple ring and we are done. Hence we may assume that R is not an Artinian ring.

We aim first at proving that in this case R is a left hereditary ring.

Let $L \neq 0$ be any left ideal, then L is a direct summand of an essential left ideal I. Let J be a non-zero two sided ideal such that $I \supset J$. Then R/J is a QF-ring. Consequently I contains a finite product of maximal two-sided ideals, and the multiplication of maximal ideals is commutative. By Lemma 1 it suffices to prove that the two sided maximal ideals of R are projective left modules, as obviously the proper prime ideals of R are maximal two sided ideals and the residue rings are Artinian rings.

Let M be any maximal ideal, then $M \neq M^2$ and R/M^2 is a QF-ring. Since R/M^2 is a local ring, it is a principal ideal ring [10]. Consequently, there results the existence of an element x in R so that $M = Rx + M^2 = xR + M^2$. We claim that x is a regular element. Because: if for some element z in R, we have $xz = 0$, then $Mz = M^2z$. Thus $Mz \subset M^n$ for every integer n and as $\bigcap_{n=1}^{\infty} M^n = 0$ we may conclude that $Mz = 0$. But R being a prime ring now yields $z = 0$. In a similar way, if $tx = 0$ then $t = 0$, and therefore x is a regular element. Let B be any non-zero two sided ideal such that $B \subset Rx$. Since R/B and all of its residue rings are QF-rings, then R/B is a principal ideal ring. As $Rx \supset B$ and as $M \supset Rx$ we have $M \supset B$. Consequently, we may assume that B is a finite product of maximal two-sided ideals, $B = M_1 \cdots M_t$, and $M_1 = M$. Let m be the smallest integer for which $Rx \supset M^mC$, where C is a finite product of maximal ideals and $M \not\supset C$. The ideal B assures the existence of the integer m. Claim: $m = 1$. If not then $m \geqq 2$, and we shall derive a contradiction. From $M = Rx + M^2$ we obtain

$$M^{m-2}C \cdot M \subset M^{m-2}C \cdot Rx + M^{m-2}C \cdot M^2 \subset Rx + M^mC \subset Rx,$$

whence $Rx \supset M^{m-1}C$. This contradiction yields the following: $M = Rx + M^2$, x is a regular element, $Rx \supset MA = AM$ and $M \not\supset A$. The projectivity of M as a left module follows from Lemma 3.

Furthermore $MA \neq 0$ and R/MA is a QF-ring as well as all of its residue rings. Therefore R/MA is a principal ideal ring. In particular, there exists an element y in R for which $M = Ry + MA \subset Ry + Rx \subset M$. Consequently M is a finitely generated left module, and by the remark to Lemma 1 so is I. Since L is a direct summand of I, we may conclude that L is a finitely generated left ideal therefore R is a left Noetherian ring.

Since in an LD-ring $\bigcap_{n=1}^{\infty} A^n = 0$ for every two-sided proper ideal, and since a left bounded, left Noetherian prime ring whose proper residue rings are QF-rings is an LD-ring we can derive the following consequences:

Corollary 6. *A left bounded prime ring R of finite left Goldie dimension, all of whose proper residue rings are QF-rings, is a left hereditary left Noetherian ring.*

Proof. If R is a simple ring, then it is an Artinian ring and we are done.

If R is not a simple ring then it is not a left Artinian ring, but it is an LD-ring. Consequently $\bigcap_{n=1}^{\infty} A^n = 0$ for every proper two-sided ideal A. The conclusion follows from Proposition 5.

THEOREM 7. *A non-Artinian left bounded ring R of finite left Goldie dimension, all of whose proper residue rings are QF-rings, is a left Noetherian, left hereditary prime ring.*

PROOF. One easily verifies that R is a left Noetherian ring whose proper prime ideals are maximal two sided ideals. Hence every proper two sided ideal contains a finite product of maximal ideals. In view of Corollary 6 it suffices to prove that R is a prime ring. Were not R a prime ring, there would result the existence of maximal two sided ideals $M_1, \cdots, M_t$ so that $M_2 \cdots M_t \neq 0$ and $M_1 M_2 \cdots M_t = 0$ Consequently R contains a minimal left ideal, and IR is a two sided ideal of finite length as a left module. Since R/IR is a QF-ring it follows that R is a left Artinian ring. This contradiction establishes the desired result.

We pass now from the symmetric condition on R—that of having its residue rings as QF-rings—to the non-symmetric condition on R—that of being an LD-ring-we obtain:

THEOREM 8. *A non-Artinian LD-ring R is a left hereditary ring.*

PROOF.* Let M be any proper ideal in R, and let x be a regular element in M^2. Let y be an element in M so that $M = Ry + M^2 = Ry + Rx$. By [9] there exists a regular element z in R so that $y - z \in Rx$. Consequently $Rz + Rx = Ry + Rx = M$, and $M = Rz + M^2$. If M is a maximal ideal in R one verifies as in the proof of Proposition 6 that $Rz \supset MA = AM$ for some ideal A in R and $M \not\supset A$. Since z is a regular element in R, Rz is a projective left module. From Lemma 3 it follows that M is a projective left module, and by Lemma 1 we may now conclude that R is a left hereditary ring.

In the commutative case it suffices to have R/M^2 a QF-ring in order to conclude on a Noetherian domain that it is a Dedekind domain [11]. The following may be regarded as a generalization of this result.

THEOREM 9. *Let R be a left Noetherian left bounded prime ring. If R/MN is a QF-ring for every pair of proper prime ideals (not necessarily distinct) M and N in R then R is a left hereditary ring.*

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* I am indebted to Professor A. V. Jategaonkar who kindly pointed out to me the way of finding the regular element z.

PROOF. Obviously every proper prime ideal in R is a maximal ideal. Let M, N be distinct maximal ideals. In the QF-ring R/MN there are only two maximal ideals, namely M and N. Therefore the radical U of R/MN is $M \cap N/MN$ and $U^2 = 0$. If $S = R/MN$ decomposes, then necessarily $R/MN \simeq R/M \oplus R/N$ whence $MN = M \cap N$. If $S = R/MN$ is an indecomposable ring then every component has length exactly two.

Since S is a QF-ring, an easy computation shows that this is impossible: $S/(M \cap N/MN) \simeq R/M \oplus R/N$ whence if e_1, e_2, e_3 are any three idempotents in S, then at least for one pair of indices i,j $(1 \leqq i \leqq j \leqq 3)$ $Se_i \simeq Se_j$. If $S = Sf_1 + \cdots, + Sf_k$ is a complete decomposition for S let $S = Sh + Sg$ where $h(g)$ is the sum of $f_{i_1} \cdots f_{i_j}$ where $Sf_{i_1} \simeq Sf_{i_l}$ for $l = 1, \cdots, j$. Then $M \cap N/MN = hSg + gSh$, $hSg \neq 0$ and $gSh \neq 0$ and thus $0 = MN/MN = hSg$ (or gSh) which is a contradiction. Therefore, for every pair of distinct maximal ideals M, N we have $MN = NM = M \cap N$, whence $R/MN = R/M \oplus R/N$. In particular, since $R/M^2(R/N^2)$ is a QF-ring also $R/(M \cap N)^2 \simeq R/M^2 \oplus R/M^2$ is a QF-ring.

Let $A \neq 0$ be any ideal in R, then necessarily A contains a product $M_1^{m_1} \cdots M_n^m$ where M_i are distinct maximal ideals and m_i suitable integers. In particular R/A is a factor ring of $R/M_1^{m_1} \cdots M_n^m \simeq R/M_1^{m_1} \oplus \cdots \oplus R/M_n^{m_n}$ whence R/A is a QF-ring. That R is a left hereditary ring is thus a consequence of Corollary 6.

Consequently, there result a kind of a converse to the result of G. Michler [7] that the proper residue rings of a bounded Dedekind ring are QF-rings.

THEOREM 10. *A bounded, Noetherian prime ring all of whose proper residue rings are QF-rings is a bounded Dedekind ring.*

PROOF. By Proposition 9 R is an hereditary ring, and being an LD-ring it has no idempotents ideals [12] thus R is a bounded Dedekind ring [7].

PROPOSITION 11. *Let l.p.dim $S \leqq 1$ for every simple left R-module S, and let R/J be a left module of finite length for every essential left ideal J in R. Then R is a left hereditary ring.*

PROOF. It suffices to prove that every essential left ideal is a projective left module. Since for an essential left ideal R/J has finite length, we may use induction on the length of R/J. Furthermore, for essential left ideals J for which the length of R/J is one, i.e. R/J is a simple left module, it is assumed that l.p.dim $R/J \leqq 1$, and thus J is a projective left module. We now assume that an essential left ideal I is projective (hence l.p.dim $R/I \leqq 1$) whenever R/I has length less than m $(m \geqq 2)$.

Let K be any essential left ideal so that the length of R/K is precisely m. If no such K exists we are done. Otherwise, let L be a proper ideal in R containing K, so that L/K is a simple left module. Since L contains K, L is an essential ideal, and R/L has length precisely $m - 1$. Consequently, l.p.dim $R/L \leqq 1$, also l.p.dim $L/K \leqq 1$, and from the exact sequence

$$0 \to L/K \to R/K \to R/L \to 0$$

it follows that l.p.dim $R/K \leqq 1$, and this completes the proof.

Obviously if R is a Noetherian ring the assumption on R/J may be replaced by the hypothesis that R/J is a left Artinian R-module.
Applying to the commutative case we obtain (see [3]).

Corollary 12. *A commutative domain whose proper prime ideals are projective is a Dedekind domain.*

Proof. Since prime ideals are projective, and since projective ideals are invertible, every prime ideal of R is finitely generated. If $P \subset Q$ are prime ideals, and Q' is the invertible ideal so that $QQ' = R$, then PQ', $Q \subset R$ and $(PQ')Q = P$ therefore proper prime ideals in R are maximal ideals. Thus R is a Noetherian domain and its proper residue rings are Artinian rings [3] and from Proposition 11 it follows that R is a Dedekind domain, (also compare with Corollary 2).

Another application to the commutative case is:

Proposition 13. *A commutative domain R, every cyclic ideal of which is a product of finitely many maximal ideals is a Dedekind domain.*

Proof. Since cyclic ideals are projective (invertible), it follows immediately that prime ideals are finitely generated projective modules. Since every ideal contains a cyclic ideal, all proper residue rings are Artinian and the result follows from Proposition 11 (also compare Corollary 12).

In [8] J. C. Robson and P. Griffith gave a short proof for the following Theorem of Asano-Michler. The following suggests an elementary proof, but we too adopt Michler's result [7] that R/M is an Artinian ring for the maximal two-sided ideals M.

Theorem 14. (*Asano-Michler*): *A bounded, Noetherian prime ring whose non-zero prime ideals are invertible, is an hereditary ring.*

Proof. Prime ideals are maximal two sided ideals (as in Corollary 12) and being invertible, every maximal ideal is a finitely generated left (right) projective module. For a maximal ideal P, R/P is an Artinian ring [7]. Since R is a left (right) bounded ring, then every essential left (right) ideal contains a finite product of maximal ideals. From Lemma 1 it follows that every essential left (right) ideal is a projective left (right) module, and this completes the proof.

In [7] Michler proves that in a bounded Dedekind prime every essential ideal can be generated by two elements, the first of which may be chosen arbitrarily as long as it is a regular element.

We aim towards a converse to this theorem. We start with the non bounded case, and then obtain the desired converse by restricting the result to the bounded case.

PROPOSITION 15. *Let R be an order in a simple Artinian ring with no proper idempotent ideals. If every essential left (and right) ideal can be generated by at most two elements the first of which may be chosen arbitrarily (as long as it is a regular element), if R/M is an Artinian ring whenever M is a maximal ideal, and if $\bigcap_{n=1}^{\infty} M^n = 0$, then the proper residue rings of R are QF-rings.*

PROOF. Let I be any proper two-sided ideal. Since I is an essential ideal, it contains a regular element x. Consequently, for every left ideal J that contains I there exists an element y in R so that $J = Rx + Ry = I + Ry$. Thus R/I is a left (similarily right) principal ideal ring. To complete the proof we have to show that R/I is an Artinian ring. Since R/M is an Artinian ring whenever M is a maximal ideal, R/M^n is an Artinian principal ideal ring for every integer n.

Let P be any proper prime ideal, and assume that $P \subsetneqq M$, then $M^n \subset M^n + P \subset M$ for every integer n. Comparing the Artinian principal ideal rings R/M^n and $R/(M^n + P)$ one easily verifies that either $M^n = M^n + P$ or else $M^{n-1} = M^n + P$. Thus, if $M^n + P \neq M^n$ then $P \neq 0$, and in the left principal ideal prime ring R/P the equality $(M/P)^{n-1} = (M/P)^n$ holds. This yields a contradiction. Therefore $P \subset M^n$ for every integer n. whence $P = O$. Hence prime ideals are maximal. Being Noetherian, every ideal I contains a finite product of prime ideals and since prime ideals are maximal ideals, R/I is an Artinian ring.

PROPOSITION 16. *Let R be a left bounded order in a simple Artinian ring. If every essential left (and right) ideal can be generated by at most two elements the first of which may be chosen arbitrarily (as long as it is a regular element), then the proper residue rings are QF-rings and R is an hereditary ring.*

PROOF. One easily verifies that R is an LD-ring, in particular R/M is an

Artinian ring whenever M is a maximal two sided ideal and $\bigcap_{n=1}^{\infty} M^n = 0$. Thus applying Proposition 15 we obtain that all proper residue rings are QF-rings. Furthermore, by Proposition 8 R is a left hereditary ring. Being a right Noetherian ring this implies that R is a right hereditary ring.

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